A ROW AND COLUMN DESIGN

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(Received : January, 1977)

INTRODUCTION

Some designs for the elimination of two way heterogeniety have been considered by many authors in the past like Srikhande [5] and Youden [6], Pearce [3] and Agarwal [1]. Youden has used the symmetrically balanced incomplete block design and Srikhande has generalised these to include the case of balanced incomplete block designs having $b=mv$ where v, b, r, k and λ are the parameters of the design and has obtained balanced designs eliminating the positional effects. Pearce [3] has given some two way elimination augmented designs. Agarwal has also constructed some designs by utilising a special class of BIB designs. Preece [4] has considered some designs for number of non-interacting treatments when applied to the same unit as a generalisation of orthogonal Latin square.

In the present paper a row and column designs have been obtained through the orthogonal partitioning of a latin square. The concept of orthogonal partitioning of latin square was given by Finney (2). He defined the orthogonal partitioning of a latin square in more general form as the partitioning of s^2 cells of a $s \times s$ latin square into k sets of sn_i $i=1,...k$) cells where $n_1+n_2 ... +n_k=s$ in such a way that the ith set has n_i cells in each row n_i cells in each column and n_i cells for each letter. Each such set is an orthogonal portion of the latin square. If the orthogonal partitioning of the latin square of size $s \times s$ is done in twogroups (s^2-s, s) and if the (s^2-s) group is considered as a row and column design with s treatments and each replicated $(s-1)$ times then this provides a row and column design of the T : T T type as classified by Pearce (1960). If in the s cells which were not considered earlier were substituted by a treatment other then the one included in the design then the designs so formed will be of the type O : T T as classified by Pearce [3]. The method of analysis for these two class of designs has been given in the present paper.

Method of Analysis

Consider a two way design with u rows and u' columns. Let there be v treatments, the ith treatment being replicated r_i times, l_{ij} denotes the number of times the ith treatment occurs in the jth row, mij' the number of times it occurs in the j'th columns, njj' is 1 or 0 according as the experiment utilises the (jj') cell or not. Ac, d will devote a matrix of c rows and d columns.

Let

$$Lv,\ u=(l_{ij}),$$
$$Mvu'=(m_{ij}')$$

and

$$Nu,\ u'=(n_{ij}'),$$
$$i=1, 2 \ldots v,$$
$$j=1, 2 \ldots u;$$
$$j'=1, 2 \ldots u'.$$

If $y_{jj'(i)}$ is the yield of the ith treatment in the jth row and j'th column, the mathematical model assumed will be

$$y_{jj}'(i)=\mu+\alpha_j+\beta_j'+\tau_i+e_{jj}' \qquad \ldots(1.1)$$

where μ is the general mean effect, α_j and β_j' are effects of the jth row and j'th column. τ_i is the effect of the ith treatment and e_{jj}' are the independently and normally distributed with mean zero and variance σ^2. Let T, R and C denote respectively the column vector of the totals of the yields of treatments, rows and column. Defining

$$n_{j.}=\sum_{j'} n_{jj}' ;$$

$$n_{.j}'=\sum_{j} n_{ij}' ;$$

$$n=\sum_{j} \sum_{j'} n_{jj}'$$

and dropping the subscripts of N, M and L we have

$$X_{11}=C-\underset{\sim}{N}' \text{ diag } \left(\frac{1}{n_{1.}}, \frac{1}{n_{2.}} \ldots \frac{1}{n_{u.}}\right) R \qquad \ldots(1.2)$$

$$X_{21}=R-\underset{\sim}{N} \text{ diag } \left(\frac{1}{n_{.1}}, \frac{1}{n_{.2}} \ldots \frac{1}{n_{.u}'}\right) C \qquad \ldots(1.3)$$

$$Y_{11}=T-\underset{\sim}{M} \text{ diag } \left(\frac{1}{n_{.1}}, \frac{1}{n_{.2}} \ldots \frac{1}{n_{.u}'}\right) C \qquad \ldots(1.4)$$

$$Y_{22}=T-\underset{\sim}{L}\ \text{diag}\left(\frac{1}{n_{1.}},\frac{1}{n_{2.}}\ \ldots\ \frac{1}{n_{u.}}\right)R \qquad \ldots(1.5)$$

$$\underset{\sim}{X}_{12}=-\underset{\sim}{M}'+\underset{\sim}{N}\ \text{diag}\left(\frac{1}{n_{1.}},\frac{1}{n_{2.}}\ \ldots\ \frac{1}{n_{u.}}\right)\underset{\sim}{L}' \qquad \ldots(1.6)$$

$$\underset{\sim}{X}_{22}=-\underset{\sim}{L}+\underset{\sim}{N}\ \text{diag}\left(\frac{1}{n_{.1}},\ \ldots\ \frac{1}{n_{.u'}}\right)\underset{\sim}{M} \qquad \ldots(1.7)$$

$$\underset{\sim}{A}_{11}=\text{diag}\,(n_{.1},n_{.2}\ \ldots\ n_{.u'})-\underset{\sim}{N}\ \text{diag}\left(\frac{1}{n_{1.}}\ \ldots\ \frac{1}{n_{u.}}\right)\underset{\sim}{N} \qquad \ldots(1.8)$$

$$\underset{\sim}{A}_{22}=\text{diag}\,(n_{1.},n_{2.},\ \ldots\ n_{u.})-\underset{\sim}{N}\ \text{diag}\left(\frac{1}{n_{.1}}\ \ldots\ \frac{1}{n_{.u'}}\right)\underset{\sim}{N}' \qquad \ldots(1.9)$$

$$\underset{\sim}{B}_{11}=\text{diag}\,(r_1,r_2\ \ldots\ r_v)-\underset{\sim}{M}\ \text{diag}\left(\frac{1}{n_{.1}}\ \ldots\ \frac{1}{n_{.u'}}\right)\underset{\sim}{M}' \qquad \ldots(1.10)$$

$$\underset{\sim}{B}_{22}=\text{diag}\,(r_1,r_2\ \ldots\ r_v)-\underset{\sim}{L}\ \text{diag}\left(\frac{1}{n_{1.}}\ \ldots\ \frac{1}{n_{u.}}\right)\underset{\sim}{L}' \qquad \ldots(1.11)$$

$$\underset{\sim}{C}_{11}=\underset{\sim}{B}_{22}-\underset{\sim}{X}'_{12}\ \underset{\sim}{A}^{*}_{11}\ \underset{\sim}{X}_{12} \qquad \ldots(1.12)$$

$$\underset{\sim}{C}_{22}=\underset{\sim}{A}_{11}-\underset{\sim}{X}_{12}\ \underset{\sim}{B}^{*}_{22}\ \underset{\sim}{X}'_{12} \qquad \ldots(1.13)$$

$$\underset{\sim}{C}_{33}=\underset{\sim}{A}_{22}-\underset{\sim}{X}_{22}\ \underset{\sim}{B}^{*}_{11}\ \underset{\sim}{X}'_{22} \qquad \ldots(1.14)$$

$$Q_{11}=Y_{22}+\underset{\sim}{X}'_{12}\ \underset{\sim}{A}^{*}_{11}\ X_{11} \qquad \ldots(1.15)$$

$$Q_{22}=X_{11}+\underset{\sim}{X}_{12}\ \underset{\sim}{B}^{*}_{22}\ Y_{22} \qquad \ldots(1.16)$$

$$Q_{33}=X_{21}+\underset{\sim}{X}_{22}\ \underset{\sim}{B}^{*}_{11}\ Y_{11} \qquad \ldots(1.17)$$

$$Q_{11}=\underset{\sim}{C}_{11}\ \hat{\tau} \qquad \ldots(1.18)$$

$$Q_{11}=\underset{\sim}{C}_{22}\ \hat{\beta} \qquad \ldots(1.19)$$

$$Q_{33}=\underset{\sim}{C}_{33}\ \hat{\alpha} \qquad \ldots(1.20)$$

where * over a matrix denotes conditional inverse and cap over letter its estimate. The above results are taken from Agrawal [1]. In any latin square of size $s \times s$ if we do the orthogonal partitioning in such a way that we remove one cell from each row, each column and also of each letter then the analysis of such type of designs will be done by the method described by Hira Lal Agrawal (1966). Treatment estimate is given by (1.18)

i.e., $$Q_{11}=\underset{\sim}{C}_{11}\ \hat{\tau}$$

Treatment sum of squares (adjusted) is obtained from $Q'_{11}\ C^*_{11}\ Q_{11}$

where $\underset{\sim}{C}^*_{11}$ is the conditional inverse of $\underset{\sim}{C}_{11}$: Q_{11} are the adjusted treatment totals ard are given by (1.15) *i.e.*,

$$Q_{11}=Y_{22}+\underset{\sim}{X}'_{12}\ \underset{\sim}{A}^*_{11}\ X_{11}$$

$\underset{\sim}{B}_{22}$ for such designs will become.

$$\underset{\sim}{B}_{22}=\text{diag}\ (r_1, r_2 \ldots r_v)-\underset{\sim}{L}\ \text{diag}\left(\frac{1}{n_{1\cdot}}\ \cdots\ \frac{1}{n_{u\cdot}}\right) L'$$

$$=(s-1)\ \underset{\sim}{I}-\frac{\underset{\sim}{L}\ \underset{\sim}{L}'}{s-1}$$

since

$$r_1=r_2=.=r_v=s-1$$

and

$$\frac{1}{n_{1\cdot}}=\frac{1}{n_{2\cdot}}-\!-=\frac{1}{n_{u\cdot}}=\frac{1}{s-1}$$

$$=(s-1)\underset{\sim}{I}-\frac{1}{s-1}\ (\underset{\sim}{I}+(s-2)\ \underset{\sim}{J})$$

$$=\frac{s\ (s-2)}{s-1}\left(\ \underset{\sim}{I}-\frac{1}{s}\ \underset{\sim}{J}\ \right) \qquad \ldots(1.21)$$

Also

$$\underset{\sim}{A}_{11}=\text{diag}\ (n_{.1},\ \ldots\ -n_{.u}')-\underset{\sim}{N}'\ \text{diag}$$

$$\left(\frac{1}{n_{1\cdot}}\cdots-\frac{1}{n_{u\cdot}}\right)\ \underset{\sim}{H}$$

$$=(s-1)\underset{\sim}{I}-\frac{1}{s-1}\ (\underset{\sim}{I}+(s-2)\ \underset{\sim}{J})$$

$$=\frac{s\ (s-2)}{s-1}\left(\underset{\sim}{I}-\frac{1}{s}\ \underset{\sim}{J}\right) \qquad \ldots(1.22)$$

Since $\underset{\sim}{I}-\frac{1}{s}\underset{\sim}{J}$ is an idempotent matrix

$$\therefore \quad \left(\underset{\sim}{I}-\frac{1}{s}\underset{\sim}{J}\right)^2=\underset{\sim}{I}-\frac{1}{s}\underset{\sim}{J}$$

$\underset{\sim}{I}$ & $\underset{\sim}{J}$ are square matrices of order $s\times s$ having their usual meanings.

$$\therefore \quad \underset{\sim}{A}^*_{11}=\frac{s-1}{s\,(s-2)}\left(\underset{\sim}{I}-\frac{1}{s}\underset{\sim}{J}\right) \qquad \ldots(1.23)$$

Now $\underset{\sim}{X}_{12}$ from (1.6) is given by

$$\underset{\sim}{X}_{12}=-\underset{\sim}{M}'+\underset{\sim}{N}'\ \text{diag}\left(\frac{1}{n_{1.}}\ \ldots\ \frac{1}{n_{u.}}\right)\underset{\sim}{L}'$$

$$=-\underset{\sim}{M}'+\frac{\underset{\sim}{N}'\,\underset{\sim}{L}'}{s-1}$$

$$=\frac{1}{s-1}\left[(s-1)\,\underset{\sim}{J}-s\,\underset{\sim}{M}'\right] \qquad \ldots(1.24)$$

$$\therefore \quad \underset{\sim}{X}'_{12}=(\underset{\sim}{X}_{12})'$$

$$=\frac{1}{s-1}\left[(s-1)\,\underset{\sim}{J}-s\,\underset{\sim}{M}\right] \qquad \ldots(1.25)$$

substituting $\underset{\sim}{A}^*_{11}$ from (1.23), $\underset{\sim}{X}_{12}$ from (1.24), $\underset{\sim}{B}_{22}$ from (1.21) and $\underset{\sim}{X}'_{12}$ from (1.25) in

$$\underset{\sim}{C}_{11}=\underset{\sim}{B}_{22}-\underset{\sim}{X}'_{12}\,\underset{\sim}{A}^*_{11}\,\underset{\sim}{X}_{12}$$

we get

$$\underset{\sim}{C}_{11}=\frac{s\,(s-2)}{s-1}\left(\underset{\sim}{I}-\frac{1}{s}\underset{\sim}{J}\right)$$

$$-\frac{1}{s-1}\left((s-1)\,\underset{\sim}{J}-s\underset{\sim}{M}\right)\left(\frac{s-1}{s\,(s-2)}(\underset{\sim}{I}-\frac{1}{s}\underset{\sim}{J})\right)\frac{1}{s-1}\left((s-1)\,\underset{\sim}{J}-s\underset{\sim}{M}'\right)$$

on simplification we get

$$\underset{\sim}{C}_{11}=\frac{s\,(s-3)}{s-2}\left(\underset{\sim}{I}-\frac{1}{s}\underset{\sim}{J}\right) \qquad (1.26)$$

$$\therefore \quad \underset{\sim}{C}^*_{11}=\frac{s-2}{s\,(s-3)}\left(\underset{\sim}{I}-\frac{1}{s}\underset{\sim}{J}\right) \qquad (1.27)$$

Hence treatment ss (adjusted)

$$=Q_{11}'\ C^{*}_{11}\ Q_{11}$$

$$=Q_{11}'\ \frac{s-2}{s\,(s-3)}\left(I-\frac{1}{s}\ J\right)Q_{11}$$

$$=\frac{s-2}{s\,(s-3)}\ Q_{11}'\ Q_{11}-\frac{s-2}{s^2\,(s-3)}\ Q_{11}'\ J\ Q_{11}$$

$$=\frac{s-2}{s\,(s-3)}\sum_{i=1}^{s}\ Q^2_{11(i)}.$$

The second part of this expression vanishes because $J\ Q_{11}=O$ and $Q_{11}\ (i)$ can be obtained from

$$Q_{11}=Y_{22}+X'_{12}\ \ A^{*}_{11}\ X_{11}$$

Substituting

$$Y_{22}=T-LR/s-1$$

$$X'_{12}=\frac{1}{s-1}\Big(\ (s-1)\ J-sM\Big)$$

$$A^{*}_{11}=\frac{s-1}{s\,(s-2)}\left(I-\frac{1}{s}\ J\ \right)$$

$$X_{11}=C-\frac{N'R}{s-1}$$

We get

$$Q_{11}=T-\frac{LR}{s-1}+\frac{1}{s\,(s-2)}\Big(\ (s-1)\ J-sM)\ \ \Big)\left(I-\frac{1}{s}\ J\right)\left(C-\frac{N'R}{s-2}\ \right)$$

on simplification we get

$$Q_{11}=T-\frac{1}{s-1}\left[L-\frac{1}{s-2}\ MN'\right]R-\frac{1}{s-2}\ M\ C$$

where

T=Column vector of treatment totals

R=column vector of row totals

C=column vector of colum totals

s=size of latin square

$\underset{\sim}{L}$=incidence matrix of treatment x rows

$\underset{\sim}{M}$=incidence matrix of treatment x coluonus

$\underset{\sim}{N}$=incidence matrix of rows x cols.

$\underset{\sim}{N}'$=Transpose of $\underset{\sim}{N}$

$$V\ (t_i - t_j)_{i \neq j} = \frac{2\ (s-2)}{s\ (s-3)} \sigma^2$$

However, when we add an extra treatment in place of those vocated cells than matrix $\underset{\sim}{L_1}$ (treatment × rows), $\underset{\sim}{M_1}$ (treatment × cols) and $\underset{\sim}{N}$ (rows × cols) can be written as

$$L_1 = \begin{bmatrix} \underset{\sim}{L} \\ J' \end{bmatrix}_{(s+1) \times s} ;$$

$$\underset{\sim}{M_1} = \begin{bmatrix} \underset{\sim}{M} \\ J' \end{bmatrix}_{(s+1) \times s}$$

$$\underset{\sim}{N} = \underset{\sim}{J}_{s \times s}$$

where $\underset{\sim}{L}$ and $\underset{\sim}{M}$ are the incidence matrices of (treat × rows) and (treatment × cols) when we do not add a supplement treatment.

Treatment ss (adjusted) is given by

$$Q'_{11}\ \underset{\sim}{C}^*_{11}\ Q_{11}$$

where C^*_{11} is the conditional inverse of the matrix C_{11}

where $$\underset{\sim}{C}_{11} = \underset{\sim}{B}_{22} - X'_{12}\ A^*_{11}\ X_{12} \qquad ...(2.1)$$

Here $$\underset{\sim}{A}_{11} = \text{diag}\ (n._1, \ldots - n._u') - \underset{\sim}{N}'\ \text{diag}\ \left(\frac{1}{n_1.} \cdots - \frac{1}{n_u.}\right) \underset{\sim}{N}$$

$$= s\ \underset{\sim}{I} - \frac{1}{s}\ \underset{\sim}{N}'\ \underset{\sim}{N}$$

$$= s\ \underset{\sim}{I} - \frac{1}{s}\ s\ \underset{\sim}{J}$$

$$= s\ \underset{\sim}{I} - \hat{J}$$

$$= s\left(\underset{\sim}{I} - \frac{1}{s}\ \underset{\sim}{J}\right) \qquad ...(2.2)$$

$$A^*_{11} = \frac{1}{s}\left(\underset{\sim}{I} - \frac{1}{s} \underset{\sim}{J} \right) \qquad \ldots(2.3)$$

Also $X_{12} = -\underset{\sim}{M}'_1 + \underset{\sim}{N}' \operatorname{diag} \left(\frac{1}{n_{1.}} \ldots\ldots \frac{1}{n_{u.}} \right) \underset{\sim}{L}'_1$

$$= -M'_1 + \frac{\underset{\sim}{N}' \underset{\sim}{L}'_1}{s}$$

$$= \left[-\underset{\sim}{M}' + \frac{s-1}{s} \underset{\sim}{J} : \underset{\sim}{O} \right] \qquad \ldots(2.4)$$

$$\underset{\sim}{B}_{22} = \operatorname{diag} (r_1 \ldots - r_v) - \underset{\sim}{L}_1 \operatorname{diag} \left(\frac{1}{n_{1.}} \ldots\ldots \frac{1}{n_{u.}} \right) L'_1$$

$$= \begin{bmatrix} (s-1) \underset{\sim}{I} & O \\ & O\ s \end{bmatrix} - \frac{1}{s} \begin{bmatrix} \underset{\sim}{L} \\ J' \end{bmatrix} \left[\underset{\sim}{L}' : J \right]$$

on simplification we get

$$\underset{\sim}{B}_{22} = \begin{bmatrix} \frac{s(s-1)-1}{s} \underset{\sim}{I} - \frac{s-2}{s} \underset{\sim}{J} & -\frac{s-1}{s} \underset{\sim}{J} \\ -\frac{s-1}{s} \underset{\sim}{J}' & s-1 \end{bmatrix} \qquad \ldots(2.5)$$

$$\therefore \underset{\sim}{X}'_{12} \; A^*_{11} \; X_{12}$$

$$= \begin{bmatrix} -\underset{\sim}{M} + \frac{s-1}{s} \underset{\sim}{J} \\ O' \end{bmatrix} \frac{1}{s} \left[I - \frac{1}{s} \underset{\sim}{J} \right] \left[-M' + \frac{s-1}{s} J : O \right]$$

on simplification we get $\underset{\sim}{X}'_{12} \; \underset{\sim}{A}^*_{11} \; \underset{\sim}{X}_{12}$

$$= \frac{1}{s} \begin{bmatrix} \underset{\sim}{I} - \frac{1}{s} \underset{\sim}{J} & 0 \\ & 0 \quad 0 \end{bmatrix} \qquad \ldots(2.6)$$

Also

$$\underset{\sim}{C}_{11} = \underset{\sim}{B}_{22} - \underset{\sim}{X}'_{12} \; A^*_{11} \; \underset{\sim}{X}_{12}$$

Substituting B_{22}, $\underset{\sim}{X}'_{12}$, $\underset{\sim}{A}^*_{11}$, $\underset{\sim}{X}_{12}$ we get

$$\underset{\sim}{C}_{11}=\begin{bmatrix} \frac{s(s-1)-2}{s}\underset{\sim}{I}+\frac{1-s(s-2)}{s^2}\underset{\sim}{J}-\frac{s-1}{s}\underset{\sim}{J} \\ -\frac{s-1}{s}\underset{\sim}{J}'. & s-1 \end{bmatrix}$$

It can be seen that

$$\underset{\sim}{C}^*_{11}=\begin{bmatrix} \frac{s}{s^2-s-2}\left[\underset{\sim}{I}-\frac{1+2s-s^2}{s(s-1)}\underset{\sim}{J}\right] & 0 \\ 0 & 0 \end{bmatrix}$$

Treatment $\underset{\cdot\cdot}{ss}$ (adjusted)

$$=Q'_{11}\ C^*_{11}\ Q_{11}$$

$$=\left[\ Q'_{11}(i), Q_{11}\,s'\right]\begin{bmatrix} \frac{s}{s^2-s-2}\left[\underset{\sim}{I}-\frac{1+2s-s^2}{s(s-1)}J\right] & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} Q_{11}(i) \\ Q_{11}\,s' \end{bmatrix}$$

$$=\frac{s}{s^2-s-2}\sum_{i=1}^{s} Q^2_{11}(i)-\frac{1+2s-s^2}{(s-1)(s^2-s-2)}Q^2_{11}\,s'$$

where $Q_{11}(i)$ are the adjusted treatment total for the original treatment and $Q_{11}\,s'$ is the adjusted treatment total for the supplemented treatment.

$$V(t_i-t_j)=\frac{2s}{s^2-s-2}\sigma^2$$
$$(i\neq j)\neq s'$$

$$V(t_i-t_s')=\frac{2s^2-3s-1}{(s-1)(s^2-s-2)}\sigma^2$$

SUMMARY

Finney [2] gave the concept of orthogonal partitioning of latin square and also gave the general definition for the partitioning into i sets. Utilising this definition the s^2 cells were partitioned into two groups as (s^2-s, s). The analysis of s treatments each treatment replicated $(s-1)$ times in s rows and s columns has been discussed. The design so formed will be of the type *T*: *TT* according to Pearce [3]. When s cells are filled with treatment other than the earlier treatments included into the design, the row and column design with $(s+1)$ treatments becomes of the type *O*: *TT*. The analysis for this

design is also discussed. The expression for the variance of different type of comparisons are also given.

ACKNOWLEDGEMENT

We are highly thankful to Dr. A. Dey, Senior Professor of Statistics and Dr. M. Singh, Scientist, I.A.S.R.I. for useful discussions while carrying out the analysis. We are also thankful to Dr. D. Singh, Director, I.A.S.R.I. for providing the necessary facilities for carrying out this investigations.

REFERENCES

[1] Agrawal, Hira Lal (1966) : "Two way elimination of heterogeneity Calcutta Statistical Association Bulletin Vol. 15. pp. 32-38.

[2] Finney, D.J. (1945) : "Some orthogonal properties of latin squares" Ann. of Engenics, Vol. 12, pp. 213-220.

[3] Pearce, S.C. (1960) : "Supplemented balance" *Biometrika* 47, pp. 263-271.

[4] Preece, D.A. (1966) : "Some balanced incomplete block designs for two sets of treatments Biometrics pp. 497-506.

[5] Srikhande S.S. (1951) : "Designs for two way elimination of heterogeneity" Annals of mathematical statistics 22 pp. 235-247.

[6] Youden, W.J. (1937) : "Use of incomplete block replications in estimating tobacco mossaic virus" contribution from Boyce Thompson Institute pp. 41-8.